One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only.

One-Sided Limits

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided**.

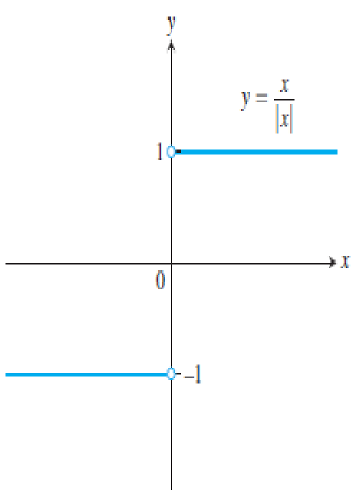


FIGURE Different right-hand and left-hand limits at the origin.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

The function $f(x) = x/|x|$ (Figure) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x \to c^+} f(x) = L.$$

The symbol "$x \to c^+$" means that we consider only values of x greater than c.

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x \to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider only x values less than c.

These informal definitions of one-sided limits are illustrated in Figure 2.25. For the function $f(x) = x/|x|$ in Figure 2.24 we have

$$\lim_{x \to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 0^-} f(x) = -1.$$

THEOREM A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

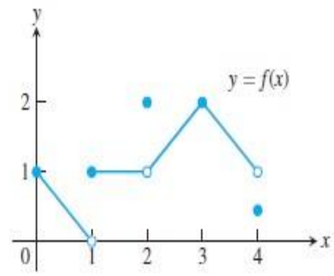


FIGURE Graph of the function

EXAMPLE For the function graphed in Figure

At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,

$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

$\lim_{x \to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

EXAMPLE Find the points at which the function f in Figure 2.35 is continuous and the points at which f is not continuous.

Solution The function f is continuous at every point in its domain $[0, 4]$ except at $x = 1, x = 2$, and $x = 4$. At these points, there are breaks in the graph. Note the relationship between the limit of f and the value of f at each point of the function's domain.

Points at which f is continuous:

At $x = 0$,	$\lim_{x \to 0^+} f(x) = f(0)$.
At $x = 3$,	$\lim_{x \to 3} f(x) = f(3)$.
At $0 < c < 4, c \neq 1, 2$,	$\lim_{x \to c} f(x) = f(c)$.

Points at which f is not continuous:

At $x = 1$,	$\lim_{x \to 1} f(x)$ does not exist.
At $x = 2$,	$\lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.
At $x = 4$,	$\lim_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.
At $c < 0, c > 4$,	these points are not in the domain of f.

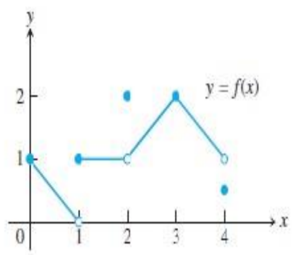


FIGURE The function is continuous on $[0, 4]$ except at $x = 1, x = 2$, and $x = 4$ (Example).

To define continuity at a point in a function's domain, we need to define continuity at an interior point (which involves a two-sided limit) and continuity at an endpoint (which involves a one-sided limit).

Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

Continuity Test

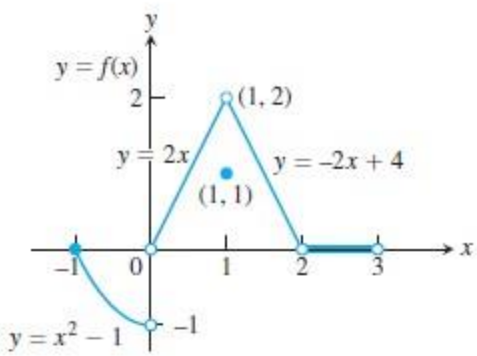
A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. **a.** Does $f(-1)$ exist?

b. Does $\lim_{x \to -1^+} f(x)$ exist?

c. Does $\lim_{x \to -1^+} f(x) = f(-1)$?

d. Is f continuous at $x = -1$?

6. **a.** Does $f(1)$ exist?

b. Does $\lim_{x \to 1} f(x)$ exist?

c. Does $\lim_{x \to 1} f(x) = f(1)$?

d. Is f continuous at $x = 1$?

7. **a.** Is f defined at $x = 2$? (Look at the definition of f.)

b. Is f continuous at $x = 2$?

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

At what points are the functions in Exercises 13–30 continuous?

13. $y = \frac{1}{x-2} - 3x$

14. $y = \frac{1}{(x+2)^2} + 4$

15. $y = \frac{x+1}{x^2 - 4x + 3}$

16. $y = \frac{x+3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \frac{1}{|x| + 1} - \frac{x^2}{2}$

19. $y = \frac{\cos x}{x}$

20. $y = \frac{x+2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \frac{\pi x}{2}$

23. $y = \frac{x \tan x}{x^2 + 1}$

24. $y = \frac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

47. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \le -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \ge 1 \end{cases}$$

continuous at every x?

48. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \le 0 \\ x^2 + 3a - b, & 0 < x \le 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?